

Minimal P -symmetric period problem of first-order autonomous Hamiltonian Systems

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Abstract: Let $P \in Sp(2n)$ satisfying $P^k = I_{2n}$, we consider the minimal P -symmetric period problem of the autonomous nonlinear Hamiltonian system

$$\dot{x}(t) = JH'(x(t)).$$

For some symplectic matrices P , we show that for any $\tau > 0$ the above Hamiltonian system possesses a $k\tau$ periodic solution x with $k\tau$ being its minimal P -symmetric period provided H satisfies the Rabinowitz's conditions on the minimal period conjecture, together with that H is convex and $H(Px) = H(x)$.

Key Words: Maslov P -index, Relative Morse index, Minimal P -symmetric period, Hamiltonian system

1 Introduction and main result

In this paper, we study the following first-order autonomous Hamiltonian system with P -boundary condition:

$$\begin{cases} \dot{x} = JH'(x), x \in \mathbb{R}^{2n} \\ x(\tau) = Px(0). \end{cases} \quad (1.1)$$

where $\tau > 0$, $P \in Sp(2n)$, and $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ is the Hamiltonian function satisfying $H(Px) = H(x)$, $\forall x \in \mathbb{R}^{2n}$. $H'(x)$ denote its gradient, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic matrix, I_n is the identity matrix on \mathbb{R}^n . Without confusion, we shall omit the subindex of the identity matrix.

A solution (τ, x) of the problem (1.1) is called a P -solution of the Hamiltonian systems. The problem (1.1) has relation with the closed geodesics on Riemannian manifold (cf.[17]) and symmetric periodic solution or the quasi-periodic solution problem (cf.[18]). In addition, the

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first author C. Liu in [22] transformed some periodic boundary problem for nonlinear delay differential systems and some nonlinear delay Hamiltonian systems to P -boundary problems of Hamiltonian systems as above, we also refer [6, 14, 16, 19] and references therein for the background of P -boundary problems in N -body problems.

Let $P \in Sp(2n)$ and $k \in \mathbb{N} = \{0, 1, 2, \dots\}$, we say P satisfies $(P)_k$ condition, if $P^k = I_{2n}$ and for each integer m with $1 \leq m \leq k-1$, $P^m \neq I$. If P satisfies $(P)_k$ condition, a P -solution (τ, x) can be extended as a $k\tau$ -periodic solution $(k\tau, x^k)$. We say that a T -periodic solution (T, x) of the Hamiltonian system in (1.1) is P -symmetric if $x(\frac{T}{k}) = Px(0)$. T is the P -symmetric period of x . T is called the minimal P -symmetric period of x if $T = \min\{\lambda > 0 \mid x(t + \frac{\lambda}{k}) = Px(t), \forall t \in \mathbb{R}\}$.

We assume the following conditions on H in our arguments:

$$(H0) \quad H \in C^1(\mathbb{R}^{2n}, \mathbb{R}) \quad \forall x \in \mathbb{R}^{2n};$$

$$(H1) \quad H \in C^2(\mathbb{R}^{2n}, \mathbb{R}) \quad \text{with } H(Px) = H(x), \quad \forall x \in \mathbb{R}^{2n};$$

$$(H2) \quad H(x) \geq 0, \quad \forall x \in \mathbb{R}^{2n};$$

$$(H3) \quad H(x) = o(|x|^2) \quad \text{as } |x| \rightarrow 0;$$

$$(H4) \quad \text{There are constants } \mu > 2 \text{ and } R_0 > 0 \text{ such that}$$

$$0 < \mu H(x) \leq (H'(x), x), \quad \forall |x| \geq R_0;$$

$$(H5) \quad H''(x) > 0, \quad \forall x \in \mathbb{R}^{2n};$$

In [35], Rabinowitz proved that the Hamiltonian system in (1.1) possesses a non-constant prescribed period solution provided H satisfying (H0) and (H2)-(H4). Because a τ/k -periodic function is also a τ -periodic function, moreover, in [35] Rabinowitz proposed a conjecture: *under the conditions (H0) and (H2)-(H4), for any $\tau > 0$, the Hamiltonian system in (1.1) possesses a τ -periodic with τ being its minimal period.* Since then, there were many papers on this minimal period problem (cf. [5], [2], [1], [10], [31], [32], [33], [9], etc.). In 1997, D. Dong and Y. Long [9] developed a new method on this prescribed minimal period solution problem and discovered the intrinsic relationship between the minimal period and the indices of a solution. Based upon the work of [9], G. Fei, Q. Qiu, T. Wang and others applied this method to various problems of Rabinowitz's conjecture (cf. [12], [13], [24], etc.). In fact, under conditions $H(0) = 0$ and $H(x) > 0, \forall x \in \mathbb{R}^{2n} \setminus \{0\}; \frac{H(x)}{|x|^2} \rightarrow +\infty$ as $x \rightarrow 0$ and $\frac{H(x)}{|x|^2} \rightarrow 0$ as $x \rightarrow +\infty$, F. Clarke and I. Ekeland proved a result on the corresponding minimal period problem for some given T in [5]; I. Ekeland and H. Hofer gave a criterion for the conjecture in [10] which is unfortunately not easy to check.

For P -boundary problem, S. Tang [28] and the first author of this paper proved that for any $0 < \tau < \frac{\pi}{\max_{t \in [0, \tau]} \|J\dot{\gamma}_P(t)\gamma_P(t)^{-1}\|}$, there exists a nonconstant P -solution with its minimal P -symmetric period $k\tau$ or $\frac{k\tau}{k+1}$ via the iteration theory of Maslov P -index. In [23], the first author

of this paper improved the result that for every $\tau > 0$, there exists a nonconstant P -solution with its minimal P -symmetric period $k\tau$ or $\frac{k\tau}{k+1}$.

For $n \in \mathbb{N}$, $k > 0$, denote by

$$Sp(2n) \equiv Sp(2n, \mathbb{R}) = \{M \in \mathcal{L}(\mathbb{R}^{2n}) \mid M^T J M = J\},$$

$$\mathcal{P}_\tau(2n) \equiv \{\gamma \in C([0, \tau], Sp(2n)) \mid \gamma(0) = I\},$$

$$Sp(2n)_k \equiv \{P \in Sp(2n) \mid P \text{ satisfies } (P)_k \text{ condition}\},$$

$$\Omega(M) \equiv \{N \in Sp(2n) \mid \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \text{ and } \nu_\lambda(N) = \nu_\lambda(M), \forall \lambda \in \sigma(M) \cap \mathbf{U}\}.$$

Denote by $\Omega^0(M)$ the path connected component of $\Omega(M)$ which contains M .

Lemma 1.1. *If $P \in Sp(2n)_k$, then there exists a matrix $I_{2p} \diamond R(\frac{2\pi}{k})^{\diamond j_1} \diamond \dots \diamond R(\frac{2r\pi}{k})^{\diamond j_r} \in \Omega^0(P^{-1})$, with $p + \sum_{m=1}^r j_m = n$.*

Proof. For $P \in Sp(2n)_k$, we have $\sigma(P^{-1}) = \sigma(P) \subseteq \{1, e^{\frac{2\pi\sqrt{-1}}{k}}, e^{\frac{4\pi\sqrt{-1}}{k}}, \dots, e^{\frac{2(k-1)\pi\sqrt{-1}}{k}}\} \subseteq \mathbf{U}$. By the Theorem 1.8.10 in [29], there exists $M_1(\omega_1) \diamond M_2(\omega_2) \diamond \dots \diamond M_s(\omega_s) \in \Omega^0(P^{-1})$ where $M_i(\omega_i)$ is a basic normal form of some eigenvalue of P^{-1} , $1 \leq i \leq s$. And the following are the basic normal forms for eigenvalues in \mathbf{U} .

$$\text{Case 1. } N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \lambda = \pm 1, b = \pm 1, 0.$$

Since $P \in Sp(2n)_k$, we have $b = 0$ and $\lambda \in \{-1, 1\} \cap \sigma(P^{-1})$.

$$\text{Case 2. } R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \theta \in (0, \pi) \cup (\pi, 2\pi).$$

Since $P \in Sp(2n)_k$, we have $\theta \in \{\frac{2\pi}{k}, \frac{4\pi}{k}, \dots, \frac{2(k-1)\pi}{k}\}$.

$$\text{Case 3. } N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \theta \in (0, \pi) \cup (\pi, 2\pi), b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, b_i \in \mathbb{R}, b_2 \neq b_3.$$

From direct computation, it is easy to check that the matrix $T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{-1}}{\sqrt{2}} \\ \frac{\sqrt{-1}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ satisfies that $TR(\theta)T^{-1} = \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix}$. Then $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} N_2(\omega, b) \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} = \begin{pmatrix} TR(\theta)T^{-1} & TbT^{-1} \\ 0 & TR(\theta)T^{-1} \end{pmatrix}$, where

$$TbT^{-1} = \begin{pmatrix} \frac{1}{2}(b_1 + b_4) - \frac{\sqrt{-1}}{2}(b_2 - b_3) & \frac{1}{2}(b_2 + b_3) - \frac{\sqrt{-1}}{2}(b_1 - b_4) \\ \frac{1}{2}(b_2 + b_3) + \frac{\sqrt{-1}}{2}(b_1 - b_4) & \frac{1}{2}(b_1 + b_4) + \frac{\sqrt{-1}}{2}(b_2 - b_3) \end{pmatrix}. \quad (1.2)$$

Denoted by

$$\begin{pmatrix} TR(i\theta)T^{-1} & X(i) \\ 0 & TR(k\theta)T^{-1} \end{pmatrix} = \begin{pmatrix} TR(\theta)T^{-1} & TbT^{-1} \\ 0 & TR(\theta)T^{-1} \end{pmatrix}^i, \quad i \in \mathbb{N} \quad (1.3)$$

where $X(i) = \begin{pmatrix} x_1(i) & x_2(i) \\ x_3(i) & x_4(i) \end{pmatrix}$ and $X(1) = TbT^{-1} = \begin{pmatrix} x_1(1) & x_2(1) \\ x_3(1) & x_4(1) \end{pmatrix}$. By direct computation, we have

$$\begin{aligned} x_1(k) &= ke^{\sqrt{-1}(k-1)\theta} x_1(1), \\ x_4(k) &= ke^{-\sqrt{-1}(k-1)\theta} x_4(1). \end{aligned} \tag{1.4}$$

Thus, from $P^k = I$ we have $X(k) = 0$, so $x_1(1) = x_4(1) = 0$, i.e.

$$\begin{aligned} \frac{1}{2}(b_1 + b_4) - \frac{\sqrt{-1}}{2}(b_2 - b_3) &= 0, \\ \frac{1}{2}(b_1 + b_4) + \frac{\sqrt{-1}}{2}(b_2 - b_3) &= 0. \end{aligned}$$

Then we have $b_2 = b_3$, which is contradict to the definition of the basic normal form $N_2(\omega, b)$.

Therefore, from (Case1)-(Case3), we get $M_i(\omega_i) = R(\theta_i)$ where $\theta_i \in \{0, \frac{2\pi}{k}, \frac{4\pi}{k}, \dots, \frac{2(k-1)\pi}{k}\}$, $1 \leq i \leq s$. And the lemma is proved. \square

For the notations in Lemma 1.1, we define

$$Sp(2n)_k(r, p; j_1, j_2, \dots, j_r) \equiv \left\{ P \in Sp(2n)_k \mid k - 2 \sum_{m=1}^r m \cdot j_m > 1, r < \frac{k}{2} \right\}.$$

Now we state the main result of this paper.

Theorem 1.1. *Suppose $P \in Sp(2n)_k(r, p; j_1, j_2, \dots, j_r)$, and the Hamiltonian function H satisfies (H1)-(H5), then for every $\tau > 0$, the system (1.1) possesses a non-constant P -solution (τ, x) such that the minimal P -symmetric period of the extended $k\tau$ -periodic solution $(k\tau, x^k)$ is $k\tau$.*

In order to prove the above result, we need to obtain the relationship between the Maslov P -index and Morse index. Thus we organize this paper as follows, in Section 2, we recall the definition and properties of the Maslov P -index theory, and we also list out the relationship between the Morse index and the Maslov P -index (see [27],[28], [23], [11] and [12]). In Section 3, we first study the iteration formula of Maslov index of paths $\xi \in \mathcal{P}_\tau(2n)$ such that $\xi(\tau) = P^{-1}$ in detail, then we will give the complete proof of the main result.

2 Preliminaries

In this section, we give a brief introduction to the Maslov P -index and its iteration properties, and then give the relationship between Maslov P -index and the relative Morse index which is studied by the first author of this paper in [23].

Maslov P -index was first studied in [7] and [21] independently for any symplectic matrix P with different treatment. The first author and S. Tang in [27, 28] defined the Maslov (P, ω) -index $(i_\omega^P(\gamma), \nu_\omega^P(\gamma))$ for any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$. And then the first author of this paper used relative index theory to develop Maslov P -index in [23] which is consistent with the definition

in [27, 28]. When the symplectic matrix $P = \text{diag}\{-I_{n-\kappa}, I_\kappa, -I_{n-\kappa}, I_\kappa\}$, $0 \leq \kappa \in \mathbb{N} \leq n$, the (P, ω) -index theory and its iteration theory were studied in [8] and then be successfully used to study the multiplicity of closed characteristics on partially symmetric convex compact hypersurfaces in \mathbb{R}^{2n} . Here we use the notions and results in [21, 27, 28].

For $\omega \in \mathbf{U}$, then the Maslov (P, ω) -index of a symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ is defined as a pair of integers(cf.[27])

$$(i_\omega^P(\gamma), \nu_\omega^P(\gamma)) \in \mathbb{Z} \times \{0, 1, \dots, 2n\},$$

where the index part

$$i_\omega^P(\gamma) = i_\omega(P^{-1}\gamma * \xi) - i_\omega(\xi), \quad (2.1)$$

$\xi \in \mathcal{P}_\tau(2n)$ such that $\xi(\tau) = P^{-1}$ and the nullity

$$\nu_\omega^P(\gamma) = \dim \ker(\gamma(\tau) - \omega P). \quad (2.2)$$

Suppose $B(t) \in C(\mathbb{R}, \mathcal{L}_s(\mathbb{R}^{2n}))$, if $\gamma \in \mathcal{P}_\tau(2n)$ is the fundamental solution of the linear Hamiltonian systems

$$\dot{y}(t) = JB(t)y, \quad y \in \mathbb{R}^{2n}, \quad (2.3)$$

we also call $(i_\omega^P(\gamma), \nu_\omega^P(\gamma))$ the Maslov (P, ω) -index of $B(t)$, denoting $(i_\omega^P(B), \nu_\omega^P(B)) = (i_\omega^P(\gamma), \nu_\omega^P(\gamma))$, just as in [21, 27, 28]. If x is a P -solution of (1.1), then the Maslov (P, ω) -index of the solution x is defined to be the Maslov (P, ω) -index of $B(t) = H''(x(t))$ and denoted by $(i_\omega^P(x), \nu_\omega^P(x))$. When $\omega = 1$, we omit the subindex, denoted by $(i^P(\gamma), \nu^P(\gamma))$ or $(i^P(B), \nu^P(B))$ for simplicity.

For $m \in \mathbb{N}$, we extend the definition of $x(t)$ which is the solution of (1.1) to $[0, +\infty)$ by

$$x(t) = P^j x(t - j\tau), \quad \forall j\tau \leq t \leq (j+1)\tau, \quad j \in \mathbb{N}$$

and define the m -th iteration x^m of x by

$$x^m = x|_{[0, m\tau]}.$$

If P satisfies $(P)_k$ condition, then x^k becomes an $k\tau$ -periodic solution of the Hamiltonian system in (1.1). We know that the fundamental solution $\gamma_x \in \mathcal{P}_\tau(2n)$ carries significant information about x . For any $\gamma \in \mathcal{P}_\tau(2n)$, S. Tang and the first author of this paper have defined the corresponding m -th iteration path $\gamma^m : [0, m\tau] \rightarrow Sp(2n)$ of γ in [27] by

$$\gamma^m(t) = \begin{cases} \gamma(t), & t \in [0, \tau], \\ P\gamma(t - \tau)P^{-1}\gamma(\tau), & t \in [\tau, 2\tau], \\ P^2\gamma(t - 2\tau)(P^{-1}\gamma(\tau))^2, & t \in [2\tau, 3\tau], \\ P^3\gamma(t - 3\tau)(P^{-1}\gamma(\tau))^3, & t \in [3\tau, 4\tau], \\ \dots\dots\dots \\ P^{m-1}\gamma(t - (m-1)\tau)(P^{-1}\gamma(\tau))^{m-1}, & t \in [(m-1)\tau, m\tau]. \end{cases} \quad (2.4)$$

If the matrix function $B(t)$ in the linear Hamiltonian system (2.3) satisfies $P^T B(t + \tau)P = B(t)$, the iteration of its fundamental solution γ is defined in the same way.

Corresponding we set

$$i_\omega^{P^m}(\gamma, m) = i_\omega^{P^m}(\gamma^m), \quad \nu_\omega^{P^m}(\gamma, m) = \nu_\omega^{P^m}(\gamma^m).$$

If the subindex $\omega = 1$, we simply write $(i^{P^m}(\gamma, m), \nu^{P^m}(\gamma, m))$, and omit the subindex 1 when there is no confusion. In the sequel, we use the notions $(i(\gamma), \nu(\gamma))$ and $(i(\gamma, m), \nu(\gamma, m))$ to denote the Maslov-type index and the iterated index of symplectic path γ with the periodic boundary condition which were introduced by Y. Long and his collaborators (cf.[25], [26], [29], [34], etc.).

In [27, 28], S. Tang and the first author of this paper obtained the important Bott-type formula and iteration inequalities for Malsov (P, ω) -index as follows.

Lemma 2.1. ([27], Bott-type iteration formula) *For any $\tau > 0$, $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbb{N}$, there hold*

$$i_{\omega_0}^{P^m}(\gamma, m) = \sum_{\omega^m = \omega_0} i_\omega^P(\gamma), \quad \nu_{\omega_0}^{P^m}(\gamma, m) = \sum_{\omega^m = \omega_0} \nu_\omega^P(\gamma), \quad (2.5)$$

where $P \in Sp(2n)$ and $\omega_0 \in \mathbf{U}$.

Lemma 2.2. ([28]) *For any path $\gamma \in \mathcal{P}_\tau(2n)$, $P \in Sp(2n)$ and $\omega \in \mathbf{U} \setminus \{1\}$, it always holds that*

$$i^P(\gamma, 1) + \nu^P(\gamma, 1) - n + i_1(\xi) - i_\omega(\xi) \leq i_\omega^P(\gamma) \leq i^P(\gamma, 1) + n - \nu_\omega^P(\gamma) + i_1(\xi) - i_\omega(\xi). \quad (2.6)$$

Lemma 2.3. ([28], iteration inequality) *For any path $\gamma \in \mathcal{P}_\tau(2n)$, $P \in Sp(2n)$ and $m \in \mathbb{N}$,*

$$\begin{aligned} & m(i^P(\gamma, 1) + \nu^P(\gamma, 1) - n) + n - \nu^P(\gamma, 1) + mi_1(\xi) - i(\xi, m) \\ & \leq i^{P^m}(\gamma, m) \\ & \leq m(i^P(\gamma, 1) + n) - n - (\nu^{P^m}(\gamma, m) - \nu^P(\gamma, 1)) + mi_1(\xi) - i(\xi, m). \end{aligned} \quad (2.7)$$

Let $e(M)$ be the elliptic height of symplectic matrix M just as the same in [29], the following lemma is important for the proof of Theorem 1.1.

Lemma 2.4 ([28]). *For any path $\gamma \in \mathcal{P}_\tau(2n)$, $P \in Sp(2n)$, set $M = \gamma(\tau)$ and extend γ to $[0, \infty)$ by (2.4). Then for any $m \in \mathbb{N}$ we have*

$$\begin{aligned} & \nu^{P^m}(\gamma, m) - \nu(\xi, 1) + \nu(\xi, m+1) - \frac{e(P^{-1}M)}{2} - \frac{e(P^{-1})}{2} \\ & \leq i^{P^{(m+1)}}(\gamma, m+1) - i^{P^m}(\gamma, m) - i^P(\gamma, 1) \\ & \leq \nu^P(\gamma, 1) - \nu^{P^{(m+1)}}(\gamma, m+1) - \nu(\xi, m) + \frac{e(P^{-1}M)}{2} + \frac{e(P^{-1})}{2}. \end{aligned} \quad (2.8)$$

Let $S_{k\tau} = \mathbb{R}/(k\tau\mathbb{Z})$, $W_P = \{z \in W^{1/2,2}(S_{k\tau}, \mathbb{R}^{2n}) \mid z(t+\tau) = Pz(t)\}$ be a closed subspace of $W^{1/2,2}(S_{k\tau}, \mathbb{R}^{2n})$. It is also a Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ as in $W^{1/2,2}(S_{k\tau}, \mathbb{R}^{2n})$. We denote by $\|\cdot\|_s$ the L^s -norm for $s \geq 1$. By the well-known Sobolev embedding theorem, we have the following embedding property: for any $s \in [1, +\infty)$, there is a constant $\alpha_s > 0$ such that

$$\|z\|_s \leq \alpha_s \|z\|, \quad \forall z \in W_P. \quad (2.9)$$

Let $\mathcal{L}_s(W_P)$ and $\mathcal{L}_c(W_P)$ denote the space of the bounded self-adjoint linear operator and compact linear operator on W_P . We define two operators $A, B \in \mathcal{L}_s(W_P)$ by the following bilinear forms:

$$\langle Ax, y \rangle = \int_0^\tau (-J\dot{x}(t), y(t))dt, \quad \langle Bx, y \rangle = \int_0^\tau (B(t)x(t), y(t))dt. \quad (2.10)$$

Suppose that $\cdots \leq \lambda_{-j} \leq \cdots \leq \lambda_{-1} < 0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots$ are all nonzero eigenvalues of A (count with multiplicity), correspondingly, e_j is the eigenvector of λ_j satisfying $\langle e_j, e_i \rangle = \delta_{ji}$. We denote the kernel of A by W_P^0 which is exactly the space $\ker_{\mathbb{R}}(P - I)$. For $m \in \mathbb{N}$, define the finite dimensional subspace of W_P by

$$W_P^m = W_m^- \oplus W_P^0 \oplus W_m^+$$

with $W_m^- = \{z \in W_P | z(t) = \sum_{j=1}^m a_{-j}e_{-j}(t), a_{-j} \in \mathbb{R}\}$ and $W_m^+ = \{z \in W_P | z(t) = \sum_{j=1}^m a_j e_j(t), a_j \in \mathbb{R}\}$. Suppose P_m is the orthogonal projections $P_m : W_P \rightarrow W_P^m$ for $m \in \mathbb{N} \cup \{0\}$. Then $\{P_m | m = 0, 1, 2, \dots\}$ is the Galerkin approximation sequence respect to A .

For a self-adjoint operator T , we denote by $M^*(T)$ the eigenspaces of T with eigenvalues belonging to $(0, +\infty)$, $\{0\}$ and $(-\infty, 0)$ with $* = +, 0$ and $* = -$, respectively. And the dimension of eigenspaces $M^*(T)$ is denoted by $m^*(T) = \dim M^*(T)$. Similarly, we denote by $M_d^*(T)$ the eigenspaces of T with eigenvalues belonging to $(d, +\infty)$, $(-d, d)$ and $(-\infty, -d)$ with $* = +, 0$ and $* = -$, respectively. we denote $m_d^*(T) = \dim M_d^*(T)$. For any adjoint operator L , we denote $L^\sharp = (L|_{ImL})^{-1}$.

The following theorem gives the relationship between the Maslov P -index and Morse index for any $P \in Sp(2n)$.

Theorem 2.1. ([23], Lemma 3.2 and Theorem 4.6) For $P \in Sp(2n)$, suppose that $B(t) \in \mathbb{C}(\mathbb{R}, \mathcal{L}_s(\mathbb{R}^{2n}))$ and $P^T B(t + \tau)P = B(t)$ with the Maslov P -index $(i^P(B), \nu^P(B))$. For any constant $0 < d \leq \frac{1}{4}\|(A - B)^\sharp\|^{-1}$, there exists an $m_0 > 0$ such that for $m \geq m_0$, there holds

$$\begin{aligned} m_d^-(P_m(A - B)P_m) &= m + i^P(B), \\ m_d^0(P_m(A - B)P_m) &= \nu^P(B), \\ m_d^+(P_m(A - B)P_m) &= m + \dim \ker_{\mathbb{R}}(P - I) - i^P(B) - \nu^P(B), \end{aligned} \quad (2.11)$$

where B is the operator defined by $B(t)$.

For the operators A and B defined in (2.10), there is another description of the Maslov P -index as follows.

Lemma 2.5 ([23]). For any two operators $B_1, B_2 \in C(\mathbb{R}, \mathcal{L}_s(2n))$ with $B_i(t + \tau) = (P^{-1})^T B_i(t) P^{-1}$, $i = 1, 2$ and $B_1 < B_2$, there holds

$$i^P(B_2) - i^P(B_1) = \sum_{s \in [0, 1]} \nu^P((1 - s)B_1 + sB_2). \quad (2.12)$$

Remark 2.1. Suppose that $B > 0$, we have

$$i^P(B) = \sum_{s \in [0, 1]} \nu^P(sB). \quad (2.13)$$

3 The proof of Theorem 1.1

In [23], the following result was proved.

Theorem 3.1. ([23]) *Suppose $P \in Sp(2n)_k$, and the Hamiltonian function H satisfies (H1)-(H4), then for every $\tau > 0$, the system (1.1) possesses a nonconstant P -solution (τ, x) satisfying*

$$\dim \ker_{\mathbb{R}}(P - I) + 2 - \nu^P(x) \leq i^P(x) \leq \dim \ker_{\mathbb{R}}(P - I) + 1. \quad (3.1)$$

Before the proof of Theorem 1.1, we need to get the information about the iteration Maslov index for paths connecting I and P^{-1} . Firstly, from Lemma 1.1 we recall that for any $P \in Sp(2n)_k$, there exists $(p, j_1, j_2, \dots, j_r) \in \mathbb{N}^{r+1}$ such that $p + \sum_{m=1}^r j_m = n$ and $I_{2p} \diamond R(\frac{2\pi}{k})^{\diamond j_1} \diamond \dots \diamond R(\frac{2r\pi}{k})^{\diamond j_r} \in \Omega^0(P^{-1})$. We remind that here $\mathbb{N} = \{0, 1, \dots\}$.

Lemma 3.1. *For $P \in Sp(2n)_k(r, p; j_1, j_2, \dots, j_r)$ and $\xi \in \mathcal{P}_\tau(2n)$ with $\xi(\tau) = P^{-1}$, there holds*

$$(k+1)i(\xi) - i(\xi, k+1) = \sum_{m=1}^r (k-2m)j_m - kp. \quad (3.2)$$

Proof. By Theorem 9.3.1 in [29](also [34]), we have

$$\begin{aligned} i(\xi, k+1) &= (k+1)(i(\xi) + S_{P^{-1}}^+(1) - C(P^{-1})) \\ &\quad + 2 \sum_{\theta \in (0, 2\pi)} E\left(\frac{(k+1)\theta}{2\pi}\right) S_{P^{-1}}^-(e^{\sqrt{-1}\theta}) - (S_{P^{-1}}^+(1) + C(P^{-1})) \\ &= (k+1)i(\xi) + kS_{P^{-1}}^+(1) - (k+2)C(P^{-1}) + 2 \sum_{\theta \in (0, 2\pi)} E\left(\frac{(k+1)\theta}{2\pi}\right) S_{P^{-1}}^-(e^{\sqrt{-1}\theta}), \end{aligned} \quad (3.3)$$

where $S_M^\pm(\omega)$ denote the splitting number of $M \in Sp(2n)$ at $\omega \in \mathbf{U}$, $C(M) = \sum_{\theta \in (0, 2\pi)} S_M^-(e^{\sqrt{-1}\theta})$ and $E(a) = \min\{m \in \mathbb{Z} \mid m \geq a\}$. One can see these notions in Chapter 9 of [29].

For $P \in Sp(2n)_k(r, p; j_1, j_2, \dots, j_r)$, $I_{2p} \diamond R(\frac{2\pi}{k})^{\diamond j_1} \diamond \dots \diamond R(\frac{2r\pi}{k})^{\diamond j_r} \in \Omega^0(P^{-1})$ with $2r < k$. By direct computation, for $\theta \in (0, \pi) \cup (\pi, 2\pi)$, we have

$$\begin{aligned} S_{P^{-1}}^+(1) &= p; \\ C(P^{-1}) &= \sum_{m=1}^r j_m; \end{aligned} \quad (3.4)$$

$$S_{P^{-1}}^-(e^{\sqrt{-1}\theta}) = \begin{cases} j_m, & \text{if } \theta = \frac{2m\pi}{k}, 1 \leq m \leq r, e^{\frac{2m\pi\sqrt{-1}}{k}} \in \sigma(P^{-1}); \\ 0, & \text{otherwise.} \end{cases} \quad (3.5)$$

So

$$\begin{aligned} \sum_{\theta \in (0, 2\pi)} E\left(\frac{(k+1)\theta}{2\pi}\right) S_{P^{-1}}^-(e^{\sqrt{-1}\theta}) &= E\left(\frac{k+1}{k}\right)j_1 + E\left(\frac{2(k+1)}{k}\right)j_2 + \dots + E\left(\frac{r(k+1)}{k}\right)j_r \\ &= 2j_1 + 3j_2 + \dots + (r+1)j_r. \end{aligned} \quad (3.6)$$

Then

$$\begin{aligned} i(\xi, k+1) &= (k+1)i(\xi) + kp - (k+2)(j_1 + j_2 + \cdots + j_r) + 2(2j_1 + 3j_2 + \cdots + (r+1)j_r) \\ &= (k+1)i(\xi) + kp - (k-2)j_1 - (k-4)j_2 - \cdots - (k-2r)j_r, \end{aligned} \quad (3.7)$$

thus $(k+1)i(\xi) - i(\xi, k+1) = \sum_{m=1}^r (k-2m)j_m - kp$. \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1.

Suppose that $(k\tau, x^k)$ is the $k\tau$ -periodic solution extended by P -solution (τ, x) in Theorem 3.1. If $k\tau$ is not the minimal P -symmetric period of $(k\tau, x^k)$, i.e., $\tau > \min\{\lambda > 0 \mid x(t+\lambda) = Px(t), \forall t \in \mathbb{R}\}$, then there exists some $l \in \mathbb{N}$ such that

$$T \equiv \frac{\tau}{l} = \min\{\lambda > 0 \mid x(t+\lambda) = Px(t), \forall t \in \mathbb{R}\}.$$

Thus $x(\tau - T) = x(0)$, both $(l-1)T$ and kT are the period of x . Since kT is the minimal P -symmetric period, we obtain $kT \leq (l-1)T$ and then $k \leq l-1$.

Note that $x|_{[0, kT]}$ is the k -th iteration of $x|_{[0, T]}$. Suppose $\gamma \in \mathcal{P}_T(2n)$ is the fundamental solution of the following linear Hamiltonian system

$$\dot{z}(t) = JB(t)z(t) \quad (3.8)$$

with $B(t) = H''(x|_{[0, T]}(t))$. Suppose ξ be any symplectic path in $\mathcal{P}_T(2n)$ such that $\xi(T) = P^{-1}$, since $P^k = I$, then

$$\nu(\xi, 1) = \nu(\xi, k+1) = \nu(\xi, l). \quad (3.9)$$

All eigenvalues of P and P^{-1} are on the unit circle, then the elliptic height

$$e(P^{-1}) = e(P) = 2n. \quad (3.10)$$

Since the system (1.1) is autonomous, we have

$$\nu_1(x|_{[0, kT]}) \geq 1 \quad \text{and} \quad \nu^{P^{l-1}}(\gamma, l-1) = \nu_1(x|_{[0, (l-1)T]}) \geq 1. \quad (3.11)$$

By Lemma 2.4, $P^{l-1} = I$ and (3.9)-(3.11), we have

$$\begin{aligned} i^I(\gamma, l-1) &= i^{P^{l-1}}(\gamma, l-1) \\ &\leq i^{P^l}(\gamma, l) - i^P(\gamma, 1) + \nu(\xi, 1) - \nu(\xi, l) + \frac{e(P^{-1}\gamma(T))}{2} + \frac{e(P^{-1})}{2} - \nu^{P^{l-1}}(\gamma, l-1) \\ &\leq i^{P^l}(\gamma, l) - i^P(\gamma, 1) + \frac{e(P^{-1}\gamma(T))}{2} + n - 1 \\ &\leq i^{P^l}(\gamma, l) - i^P(\gamma, 1) + 2n - 1. \end{aligned} \quad (3.12)$$

Note that $i^{P^l}(\gamma, l) = i^P(x|_{[0, \tau]}) \leq \dim \ker_{\mathbb{R}}(P - I_{2n}) + 1$, here we write $i^P(x|_{[0, \tau]})$ for $i^P(x)$ to remind the solution x is defined in the interval $[0, \tau]$. By the definition of Maslov P -index,

$$i^I(\gamma, l-1) = i_1(\gamma, l-1) + n.$$

So we get

$$i_1(\gamma, l-1) \leq \dim \ker_{\mathbb{R}}(P-I) - i^P(\gamma, 1) + n. \quad (3.13)$$

By the condition (H5) and Remark 2.1, we have

$$i^P(\gamma, 1) = i^P(B) = \sum_{s \in [0,1)} \nu^P(sB) = \sum_{s \in [0,1)} \dim \ker_{\mathbb{R}}(\gamma_B(sT) - P). \quad (3.14)$$

Here we remind that $B(t)$ and γ_B are defined in (3.8). Since $\gamma_B(0) = I$, so $\dim \ker_{\mathbb{R}}(\gamma_B(sT) - P) = \dim \ker_{\mathbb{R}}(P - I)$ when $s = 0$. Thus we have

$$i^P(\gamma, 1) \geq \dim \ker_{\mathbb{R}}(P - I). \quad (3.15)$$

From (3.12), it implies

$$i_1(\gamma, l-1) \leq n. \quad (3.16)$$

By the convex condition (H5), we also have

$$i_1(x|_{[0,kT]}) \geq n \quad \text{and} \quad i_1(x|_{[0,(l-1)T]}) \geq n. \quad (3.17)$$

We set $m = \frac{l-1}{k}$. Note that $x|_{[0,(l-1)T]}$ is the m -th iteration of $x|_{[0,kT]}$. By (3.11), (3.16), (3.17) and Lemma 4.1 in [26], we obtain $m = 1$ and then $k = l-1$. From the above process (3.12)-(3.13) and (3.15)-(3.17), we obtain $k = l-1$ provided $e(P^{-1}\gamma(T)) = 2n$, and

$$\begin{aligned} i^P(\gamma, 1) &= \dim \ker_{\mathbb{R}}(P - I); \\ i^P(\gamma, l) &= i^P(\gamma, k+1) = \dim \ker_{\mathbb{R}}(P - I) + 1, \\ \nu^{P^k}(\gamma, k) &= \nu^P(\gamma, 1) = 1. \end{aligned} \quad (3.18)$$

Here we remind that the left inequality in (2.7) of Lemma 2.3 holds independent of the choice of $\xi \in \mathcal{P}_{\tau}(2n)$, then for any $\xi \in \mathcal{P}_{\tau}(2n)$ we have

$$i^P(\gamma, k+1) \geq (k+1)(i^P(\gamma, 1) + \nu^P(\gamma, 1) - n) + n - 1 + (k+1)i_1(\xi) - i(\xi, k+1). \quad (3.19)$$

By the condition $P \in Sp(2n)_k(r, p; j_1, j_2, \dots, j_r)$, we get

$$\dim \ker_{\mathbb{R}}(P - I) = 2p, \quad (3.20)$$

$$k - 2 \sum_{m=1}^r m \cdot j_m > 1. \quad (3.21)$$

Applying (3.18), (3.20) and Lemma 3.1 to (3.19), we get

$$k - 2 \sum_{m=1}^r m \cdot j_m \leq 1. \quad (3.22)$$

It is contradict to the inequality (3.21). So the minimal P -symmetric period of $(k\tau, x^k)$ is $k\tau$.

Remark 3.1. Note that $e(P^{-1}\gamma(T)) = e((P^{-1}\gamma(T))^l) = e(P^{-1}\gamma(\tau)) = 2n$ is required in the above proof. If $e(P^{-1}\gamma(T)) \leq 2n - 2$, we get $i_1(\gamma, l-1) < n$ by taking the same process as (3.12)-(3.13). It contradicts to the second inequality of (3.17). At this moment, the minimal P -symmetric period of $(k\tau, x^k)$ is $k\tau$.

The condition (H5) can be replaced by a weaker condition: $H''(x(t)) \geq 0$ and $\int_0^{\tau} H''(x(t))dt > 0$ for the P -solutuin (τ, x) in Theorem 3.1.

References

- [1] A.Ambrosetti, V.Coti Zelati, *Solutions with minimal period for Hamiltonian systems in a potential well*, Ann. Inst. H. Poincaré Anal. Non Linéaire **4** (1987), 242–275.
- [2] A.Ambrosetti, G.Mancini, *Solutions of minimal period for a class of convex Hamiltonian systems*, Math. Ann. **255** (1981), 405–421.
- [3] K. C.Chang, *Infinite dimensional Morse theory and multiple solution problems*, in Progress in Nonlinear Differentiation Equations and Their Application, Vol.6, (1993).
- [4] K. C. Chang, J. Liu and M. Liu, *Nontrivial periodic solutions for strong resonance Hamiltonian systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14**(1) (1997), 103–117.
- [5] F. Clarke, I. Ekeland, *Hamiltonian trajectories having prescribed minimal period*, Comm. Pure. Appl. Math. **33** (1980), no.3, 103–116.
- [6] A.Chenciner and R.Montgomery, *A remarkable periodic solution of the three body problem in the case of equal masses*, Ann. of Math. **152**: 3 (2000), 881–901.
- [7] Y. Dong, *P-index theory for linear Hamiltonian systems and multiple solutions for nonlinear Hamiltonian systems*, Nonlinearity **19**: 6 (2006), 1275–1294.
- [8] Y. Dong and Y. Long, *Closed characteristics on partially symmetric compact convex hypersurfaces in R^{2n}* , J. Diff. Equa. **196** (2004), 226–248.
- [9] D. Dong and Y. Long, *The iteration formula of the Maslov-type index theory with applications to nonlinear Hamiltonian systems*, Trans. Amer. Math. Soc. **349** (1997), 2619–2661.
- [10] I. Ekeland, H. Hofer, *Periodic solutions with prescribed minimal period for convex autonomous Hamiltonian systems*, Invent. Math. **81** (1985), 155–188.
- [11] G.Fei, *Relative Morse index and its application to Hamiltonian systems in the presence of symmetries*, J. Diff. Equa. **122** (1995), 302–315.
- [12] G.Fei and Q.Qiu, *Minimal period solutions of nonlinear Hamiltonian systems*, Nonlinear Analysis, Theory, Method Applications **27**: 7 (1996), 821–839.
- [13] G.Fei, S. K.Kim, T.Wang *Minimal period estimates of period solutions for superquadratic Hamiltonian systems*, J. Math. Anal. Appl. **238** (1999), 216–233.
- [14] D. Ferrario and S. Terracini, *On the existence of collisionless equivariant minimizers for the classical n-body problem*, Invent. Math. **155**: 2 (2004), 305–362.
- [15] N.Ghoussoub, *Location, multiplicity and Morse indices of min-max critical points*, J. Reine Angew Math. **417** (1991), 27–76.

- [16] X.Hu and P.Wang, *Conditional Fredholm determinant for the S -periodic orbits in Hamiltonian systems*, Journal of Functional Analysis **261** (2011), 3247–3278.
- [17] X.Hu and S.Sun, *Index and stability of symmetric periodic orbits in Hamiltonian systems with application to figure-eight orbit*, Comm. Math. Phys. **290** (2009), 737–777.
- [18] X. Hu and S. Sun, *Morse index and the stability of closed geodesics*, Sci. China Math. **53**: 5 (2010), 1207–1212.
- [19] X.Hu and S.Sun, *Stability of relative equilibria and Morse index of central configurations*, C. R. Acad. Sci. Paris **347** (2009), 1309–1312.
- [20] A.Lazer and S.Solomini, *Nontrivial solution of operator equations and Morse indices of critical points of min-max type*, Nonlinear Anal. **12** (1988), 761–775.
- [21] C.Liu, *Maslov P -index theory for a symplectic path with applications*, Chin. Ann. Math. **4** (2006), 441–458.
- [22] C.Liu, *Periodic solutions of asymptotically linear delay differential systems via Hamiltonian systems*, J. Diff. Equa. **252** (2012), 5712–5734.
- [23] C. Liu, *Relative index theories and applications*, Top. Meth. Nonl. Anal, to appear.
- [24] C. Liu, *Minimal period estimates for brake orbits of nonlinear symmetric Hamiltonian systems*, Discrete and continuous dynamical systems **27**(1) (2010), 337–355.
- [25] C.Liu and Y.Long, *An optimal increasing estimate of the iterated Maslov-type indices*, Chinese Sci. Bull. **42** (1997), 2275–2277.
- [26] C.Liu and Y.Long, *Iteration inequalities of the Maslov-type index theory with applications*, J. Diff. Equa. **165** (2000), 355–376.
- [27] C.Liu and S.Tang, *Maslov (P, ω) -index theory for symplectic paths*, Advanced Nonlinear Studies **15** (2015), 963–990.
- [28] C.Liu and S.Tang, *Iteration inequalities of the Maslov P -index theory with applications*, Nonlinear Analysis **127** (2015), 215–234.
- [29] Y.Long, *Index theory for symplectic paths with application*, Progress in Mathematics, Vol. 207, Birkhäuser Verlag, 2002.
- [30] Y.Long, *Bott formula of the Maslov-type index theory*, Pacific J. Math. **187** (1999), 113–149.
- [31] Y. Long, *The minimal period problem for classical Hamiltonian systems with even potentials*, Ann. Inst. H. Poincaré Anal. non. linéaire **10**(1993), 605–626.
- [32] Y. Long, *The minimal period problem of periodic solutions for autonomous superquadratic second order Hamiltonian systems*, J. Diff. Equa. **111**(1994), 147–174.

- [33] Y. Long, *On the minimal period for periodic solutions of nonlinear Hamiltonian systmes*, Chinese Ann. of Math. **18B**(1997), 481–484.
- [34] Y.Long and C.Zhu, *Closed characteristics on compact convex hypersurfaces in R^{2n}* , Annals of Math. **155** (2002), 317–368.
- [35] P. H.Rabinowitz, *Periodic solutions of Hamiltonian systmes*, Comm. Pure Appl. Math. **31** (1978), 157–184.
- [36] P. H.Rabinowitz, *Minimax method in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math., No.65, A. M. S., Providence(1986).
- [37] S.Solimini, *Morse index estimates in min-max theorems*, Manuscript Math. **63** (1989), 421–453.